Collective dynamics: From agent-based models to PDE systems

Susana Gomes (University of Warwick)

10/10/2024 – WMS Talks - Warwick Mathematics Society

WARWICK

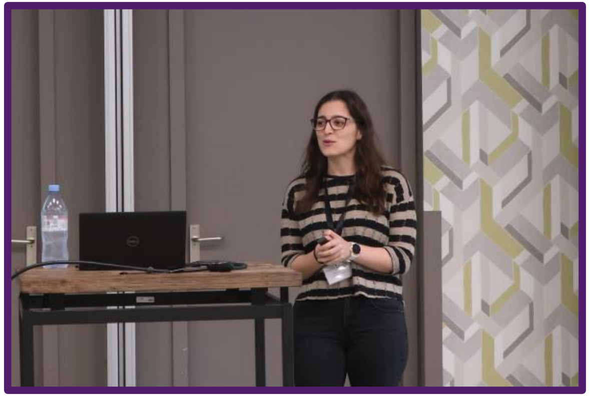
What I'll talk about today…

- A bit about me and my journey
- A brief introduction to collective dynamics, and different modelling approaches, including when each one is more appropriate
- Some examples of problems I have worked on in this context
- A short look at other (applied maths) problems I have worked on (and maybe some I haven't but would like to!)

A bit about me

WARWICK
THE UNIVERSITY OF WARWICK

- Originally from Porto (Portugal)
- Moved to Coimbra for an **Undergraduate degree in Mathematics**, and **MSc in Applied Analysis and Computational Mathematics**.
- Moved to Imperial College London (UK) in 2012 for a **PhD in Applied Mathematics and Mathematical Physics** (and postdocs).
- Moved to the University of Warwick in 2018 where now I am an Associate Professor.



Picture: me giving a talk at a conference in the Netherlands earlier this year

A bit about being a lecturer

WARWICK
THE UNIVERSITY OF WARWICK

My job is divided into a lot of parts…

- The part you see: Teaching undergraduate and MSc students
    - Includes tutorials and "normal" lectures, exam boards, sometimes teaching in other departments.
    - what I was the most afraid of, but the most fun!
- The "main" part: Research in Applied Mathematics
    - writing papers and going to conferences to give talks,
    - supervising MSc and PhD students,
    - writing grant proposals,
    - reviewing papers.
- Other university things.
    - Being in committees, helping with open days, attending graduation ceremonies, organising events, …
- Mentoring and tutoring students and younger researchers (one of my favourite parts!)

A bit about my research (in general)

WARWICK
THE UNIVERSITY OF WARWICK

I work in all sorts of areas of Applied Mathematics, but my main interests include:

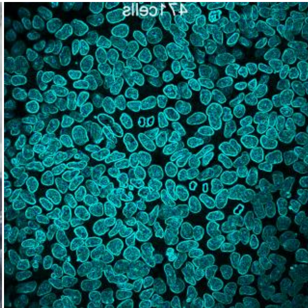
- **Control** theory (study how to manipulate systems to achieve some desired outcome)
  - Can talk a bit about this at the end, if there is time.
- Real-world applications (physical, life, and social sciences)
  - I will show you a couple of these today.
- Modelling with **differential equations** (ODEs, SDEs, PDEs), recently also using **networks**
  - Most of today will be about ODEs and SDEs.
- In some cases, I also solve **inverse problems,** in particular parameter estimation from real-world data.

Collective dynamics

Interacting particle systems are ubiquitous in the real-world, appearing in several application areas

- Biology and Life Sciences
  - Flocks of birds, schools of fish, herds od sheep, …
  - Cell dynamics
- Social Sciences
  - crowd dynamics
  - opinion dynamics, ...
- Physics and Engineering
  - Drones, robots, …
  - Molecular dynamics
  - Movement of galaxies
- Many other examples



Modelling approaches

WARWICK
THE UNIVERSITY OF WARWICK

Depending on the application, or what we want to be able to say from our models, we can take different approaches to model collective behaviour.

Some examples I will talk about include:

- One-to-one interactions (agent-based models)
  - How (and when) pairs of individuals interact *matters*
- Individual interactions within a group (interacting particle systems)
  - All pairs of individuals interact in a “similar” way, and it doesn’t matter which specific pair is interacting
- Modelling the group as a whole (macroscopic models)
  - We care mostly about the whole group behaviour, rather than individual interactions.

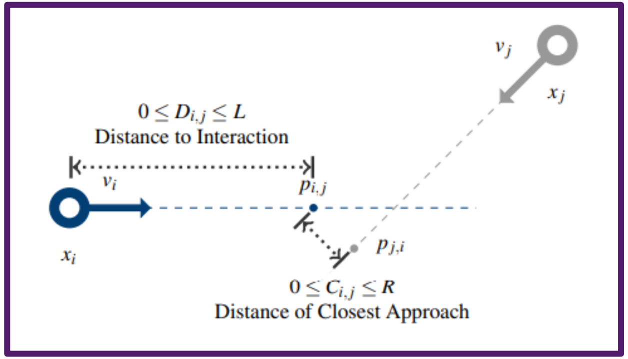
So… How do these models look like?

Agent-based models

WARWICK
THE UNIVERSITY OF WARWICK

These models are used when we want to describe interactions accurately, usually resulting from a *rational behaviour* point of view. Who each agent interacts with matters.

Example 1: Two pedestrians interacting to avoid collisions
In the figure, pedestrians at positions x_i, x_j with velocity v_i, v_j see each other and update their path to avoid a collision.



Example 2: Two people interact and change their opinion based on their conversation (with more detail later)

Figure from Bailo, Carrillo, and Degond (2018), “Pedestrian models based on rational behaviour”, in Crowd Dynamics, Vol. 1

Agent-based models

WARWICK
THE UNIVERSITY OF WARWICK

They are based on one-to one interactions, usually modelled by a Markov process (or Markov chain). At each time step, we

1. Select what individual to consider
2. Select who they will interact with
3. Update the state of one (or both) individuals according to some rule.

e.g. in **opinion dynamics**, after choosing individual i, who has opinion x_i, they will choose to interact with person j, with opinion x_j, with probability p_{ij}. After interacting, they will update their opinion by averaging their opinions out *if* their opinions are close enough, e.g., if

$$d(x_i, x_j) \leq R.$$

Interacting particle systems

WARWICK
THE UNIVERSITY OF WARWICK

These models are more appropriate for a population of agents of the “same type”. This means that everyone behaves in the same way and interacts at the same rate.

Example 1: Cell dynamics: one (or multiple) population(s) of cells interact in some experimental environment. The cell(s) can interact via attraction or repulsion, or other interesting dynamics.

Example 2: Birds flying in a group, with interactions via what is visible to them. Can exhibit very interesting behaviour, such as flocking, or milling.

Interacting particle systems

IPS are systems of (ordinary or stochastic) differential equations which usually include an "environment" term, and an interaction term. If they are an SDE, they also include noise:

$$\frac{dX_i(t)}{dt} = V(X_i(t)) + \frac{1}{N}\sum_{j\neq i}^{N} K(X_i(t) - X_j(t)) + \sigma \cdot \xi^i(t), \qquad i = 1, \dots, N.$$

- $V(X_i(t))$: How each agent interacts with the environment
- $\frac{1}{N}\sum_{j\neq i}^{N} K(X_i(t) - X_j(t))$: How each agent interacts with other agents
- $\sigma \cdot \xi^i(t)$: White noise

Interacting particle systems

These models can also have more than one variable. For example, in **pedestrian dynamics**, we can consider a pedestrian's position *and* velocity.

$$dX_i = V_i\, dt,$$
$$dV_i = F(X_i)dt + G(X_1, \dots, X_N, V_1, \dots, V_N)dt + \sigma dB_t^i.$$

When $F(x)$ has information about the environment and $G(x_1, \dots, x_N, v_1, \dots, v_N)$ includes attraction and repulsion forces, this is called the **social force model** and was proposed by Helbing.

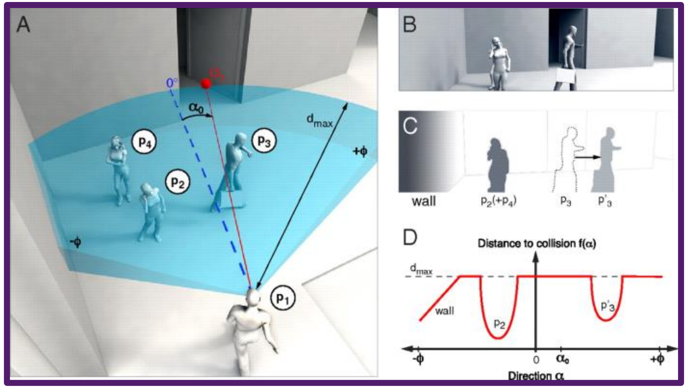


Figure from Moussaid, Helbing, and Theraulaz (2011) PNAS

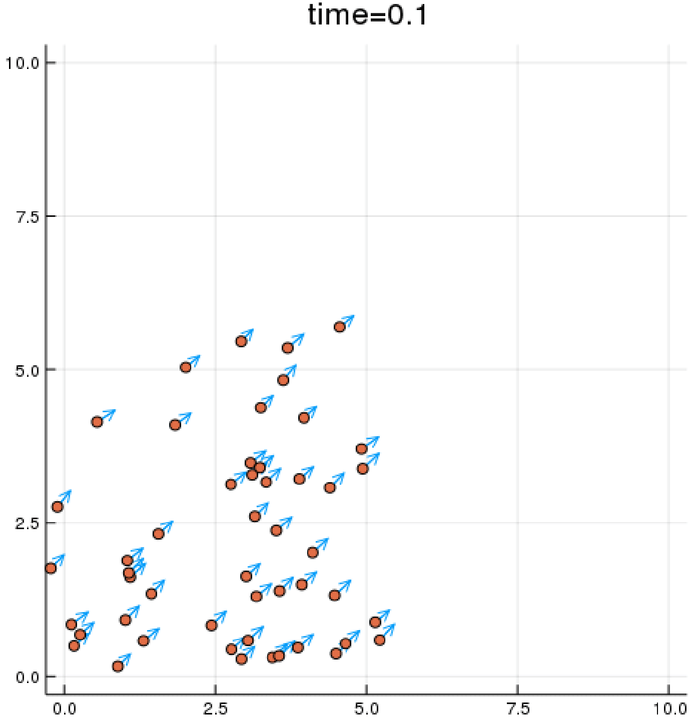
Interacting particle systems

Similarly, for flocks of birds, we can consider the **Cucker-Smale model**, which is a system of ODEs in 2D and can be written as:

$$\frac{dX_i}{dt} = V_i,$$

$$\frac{dV_i}{dt} = \frac{\lambda}{N}\sum_{j=1}^{N} \psi(\|X_j - X_i\|)\,(v_j - v_i)$$

with $\psi(r) = \frac{1}{(1+r^2)^{\beta}}$.



Mean-field limits

WARWICK
THE UNIVERSITY OF WARWICK

These models are only appropriate for the *large population limit* of an interacting particle system. They are usually an equation for a quantity of interest, like the density.

Example 1: The number of people who disagree with something, or the number of people infected with a disease.

Example 2: Large groups of pedestrians (crowds) moving in a shared space (e.g. a corridor). When modelling two groups, we sometimes observe lane formation.

Figure from Smithsonian Magazine.

Mean-field limits

The most common thing to do for mean-field limits, is to consider the *empirical density* of the particles. This is given by

$$\rho_N(x,t) = \frac{1}{N}\sum_{i=1}^{N} \delta(x - X_i(t)) \, .$$

This function approximates the probability of finding a particle at the position x at time t. Plugging this into the equation, taking $N \to \infty$, and doing some analysis, we can obtain a *partial differential equation,* called the **Fokker-Planck equation** for the density $\rho(x,t)$:

$$\frac{\partial \rho}{\partial t} = \frac{\sigma^2}{2}\frac{\partial^2 \rho}{\partial x^2} - \frac{\partial}{\partial x}\left(\rho(x,t)V(x) + \rho(x,t)\int K(x-y)\rho(y)dy\right).$$

And this can help make predictions on the behaviour of the population.

Recall that $\frac{dX_i(t)}{dt} = V(X_i(t)) + \frac{1}{N}\sum_{j\neq i}^{N} K(X_i(t) - X_j(t)) + \sigma \cdot \xi^i(t)$

Example: Pedestrian dynamics

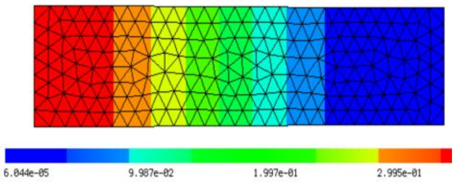
For a simple SDE model for the trajectories,

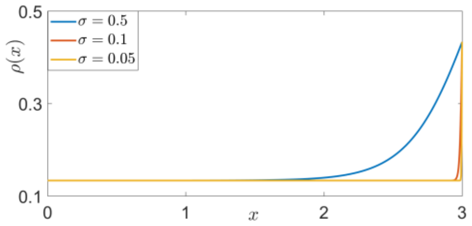
$$dX_i(t) = F\big(\rho(X_t)\big)dt + \sqrt{2\Sigma}dB_i(t),$$

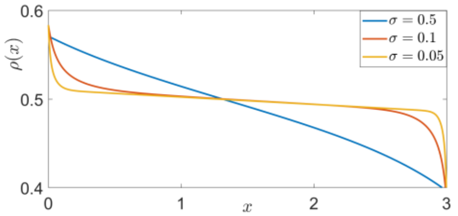
the Fokker-Planck equation is

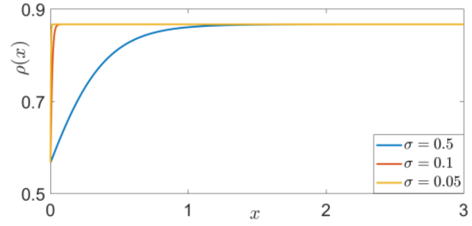
$$\frac{\partial \rho}{\partial t} = \nabla \cdot (\Sigma \nabla \rho + \rho F(\rho)).$$

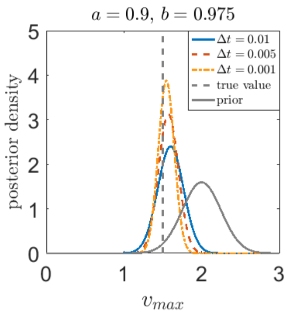
- We can use it to predict the long time behaviour of crowds because we can find *steady states*.
- We can use trajectories from experiments to estimate the maximum speed of pedestrians.
- We can also do this for irregular domains and we are working on making our models more general.

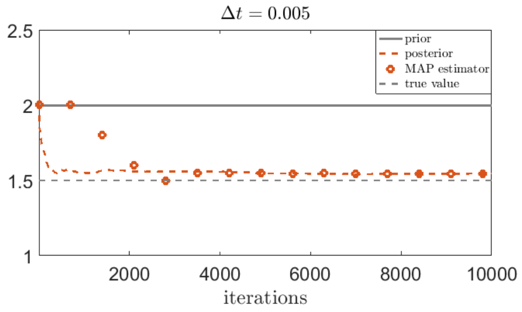












Example: Pedestrian dynamics

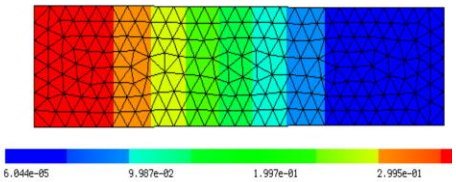
For a simple SDE model for the trajectories,

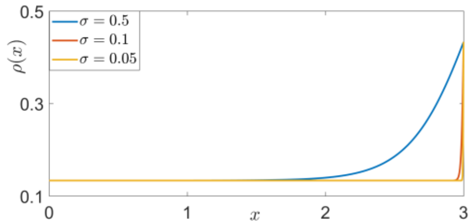
$$dX_i(t) = F\big(\rho(X_t)\big)dt + \sqrt{2\Sigma}dB_i(t),$$

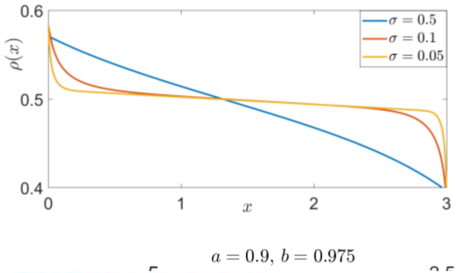
the Fokker-Planck equation is

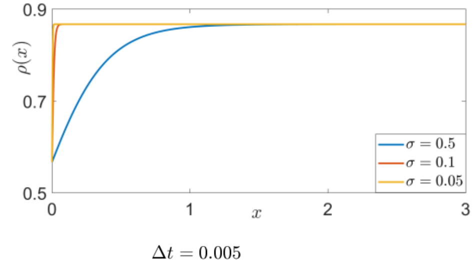
$$\frac{\partial \rho}{\partial t} = \nabla \cdot (\Sigma \nabla \rho + \rho F(\rho)).$$

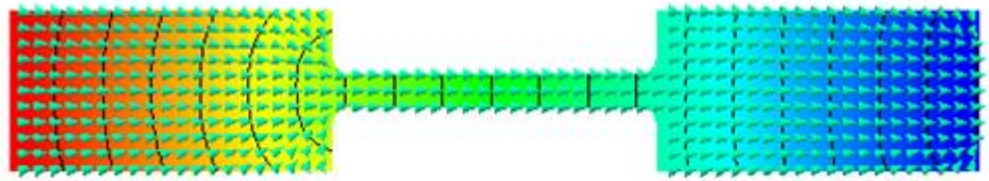
- We can use it to predict the long time behaviour of crowds because we can find *steady states*.
- We can use trajectories from experiments to estimate the maximum speed of pedestrians.
- We can also do this for irregular domains and we are working on making our models more general.











Putting some of these things together using scaling limits

(example: opinion dynamics)

Agent-based model

Now we can look at an example with more detail. The algorithm is as follows:

1. Choose two individuals i and j uniformly at random with replacement.

2. They interact with probability $p_{ij}(x)$, with person i updating their opinion according to:

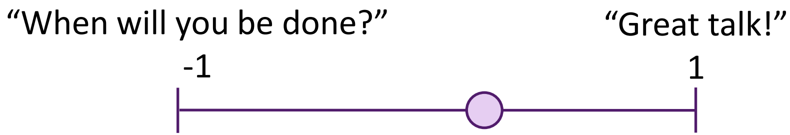
$$x_i(t+h) = \begin{cases} x_i(t) + \mu^h \left(x_j(t) - x_i(t)\right) & \text{with probability } p_{ij}(x) \\ x_i(t) & \text{with probability } 1 - p_{ij}(x)\,. \end{cases}$$

3. Repeating until time T is reached.

Markov Processes

This is a Markov process, which means the next state of the system only depends on the current state, i.e., the system has no memory.

To simulate this process, we simply need to define μ^h and $p_{ij}(x)$.



We define $\mu^h = Nh$, and $p_{ij}(x) = \phi(|x_j - x_i|)$, where $\phi(r)$ is the interaction function. Common choices include a bounded confidence function:

$$\phi(|x_j - x_i|) = \begin{cases} 1, & if\ |x_j - x_i| \le R, \\ 0, & otherwise. \end{cases}$$

Convergence to an ODE or SDE model

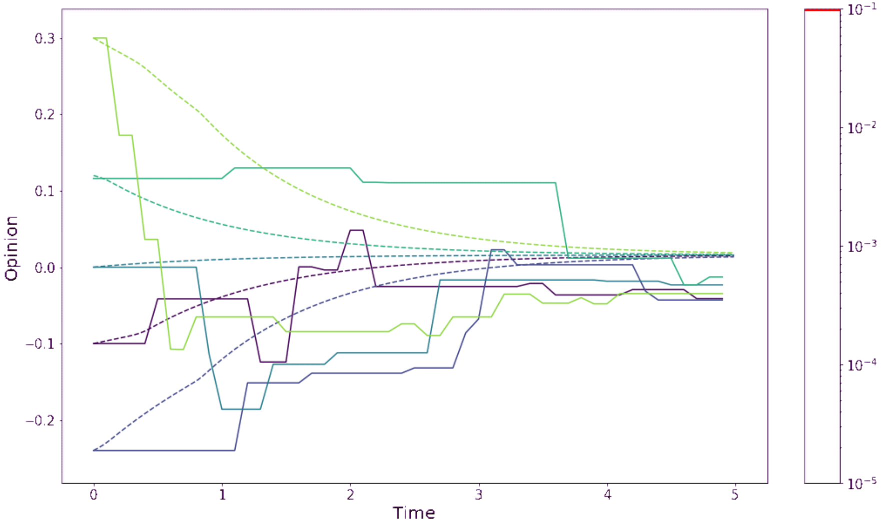
Under the right conditions ($\mu^h = Nh$), it can be shown that this agent-based model *converges* to an ODE model, known as the Hegselmann-Krause model.

This model looks like

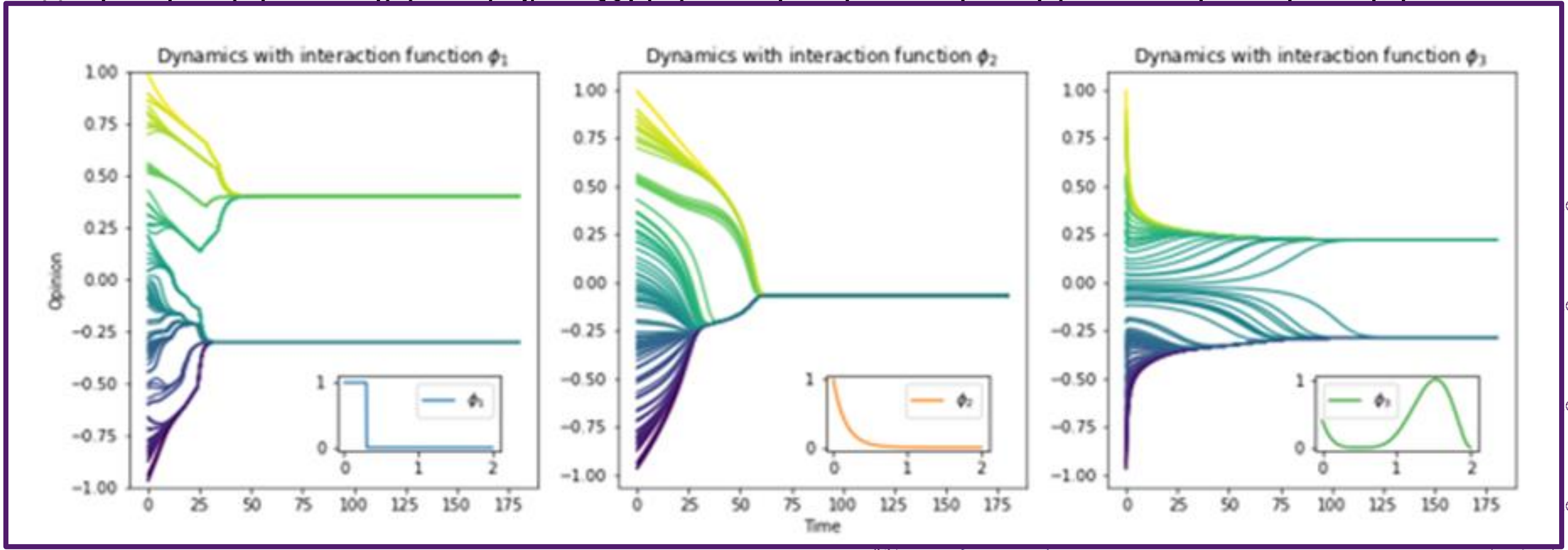
$$\frac{dx_i}{dt} = \frac{1}{N}\sum_{j=1}^{N} \phi(|x_j - x_i|)(x_j - x_i),$$

And it is easy to see that $\phi(|x|)$ can measure:

- The attention you give to someone's opinion
- How much an opinion influences you
- How much discomfort an opinion creates
- A mix of these effects



Convergence to an ODE or SDE model



- A mix of these effects

Large population limit

Some people do research on opinion dynamics in the mean-field limit. There are a few versions of the model, but a recent one would consider a PDE for the probability of someone having opinion x at time t (assuming the HK model has a noise term):

$$\frac{\partial \rho}{\partial t} = \frac{\partial}{\partial x}\left(\rho(x,t)\int (x-y)\rho(y,t)\phi(x-y)\,dy\right).$$

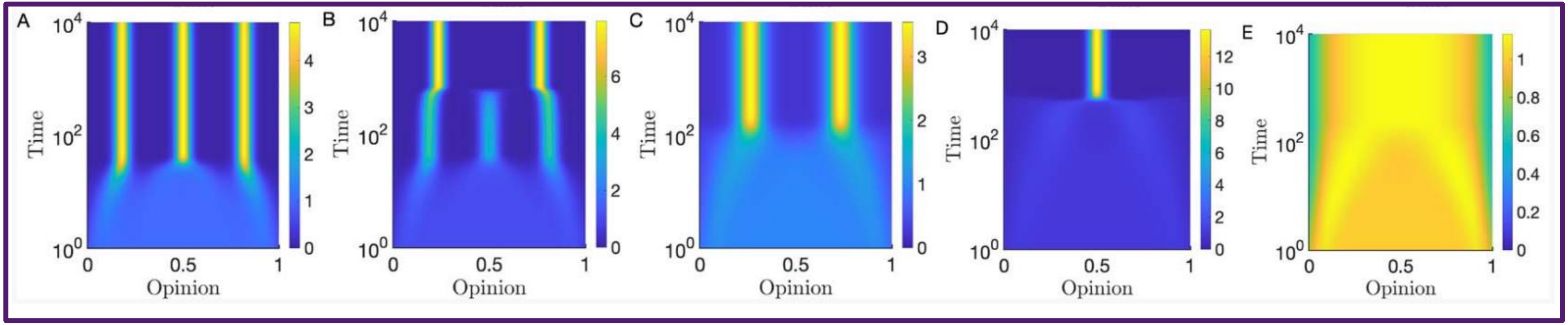


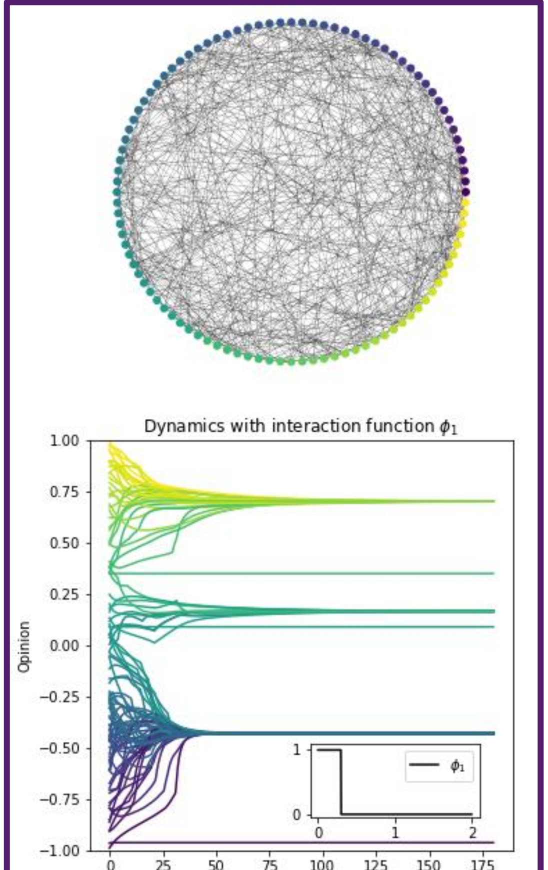
Figure from Goddard et al (2022), "Noisy bounded confidence models for opinion dynamics"

What else can we do?

A more realistic (and very interesting!) thing to do is to add a network to the model.

This introduces relationships between different agents.

- We use **graph theory** to modify the equations.

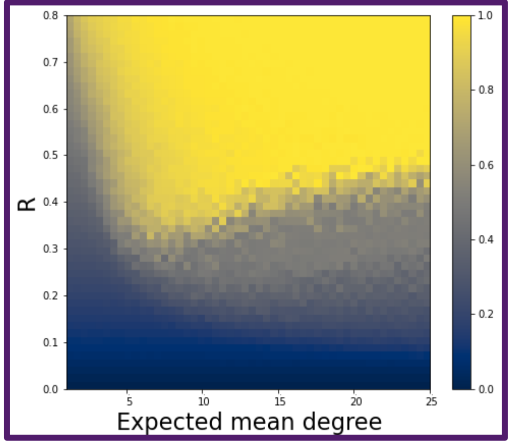


What else can we do?

A more realistic (and very interesting!) thing to do is to add a network to the model.

This introduces relationships between different agents.

- We use **graph theory** to modify the equations.
- We can study how different networks and different interaction radius affect the dynamics.

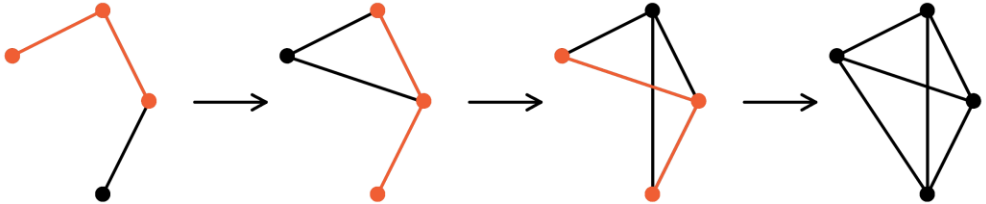


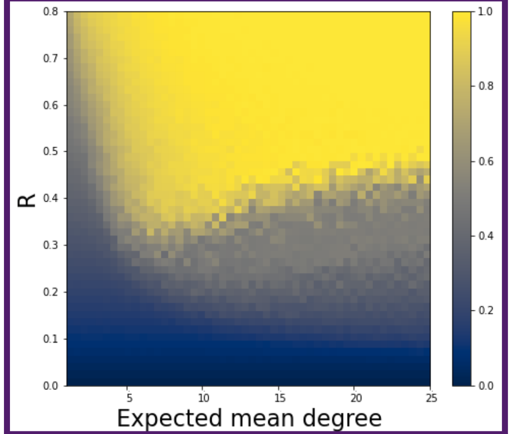
What else can we do?

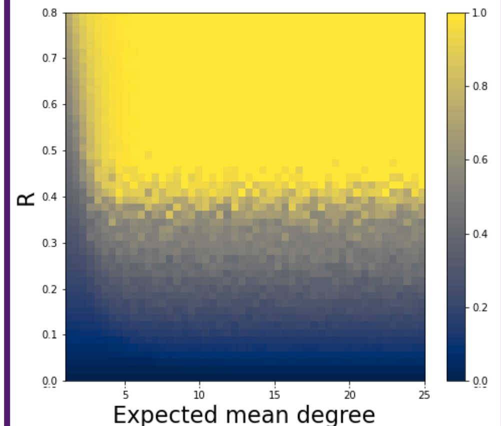
A more realistic (and very interesting!) thing to do is to add a network to the model.

This introduces relationships between different agents.

- We use **graph theory** to modify the equations.
- We can study how different networks and different interaction radius affect the dynamics.
- We can also make the networks change over time (for example, using friend of a friend dynamics)



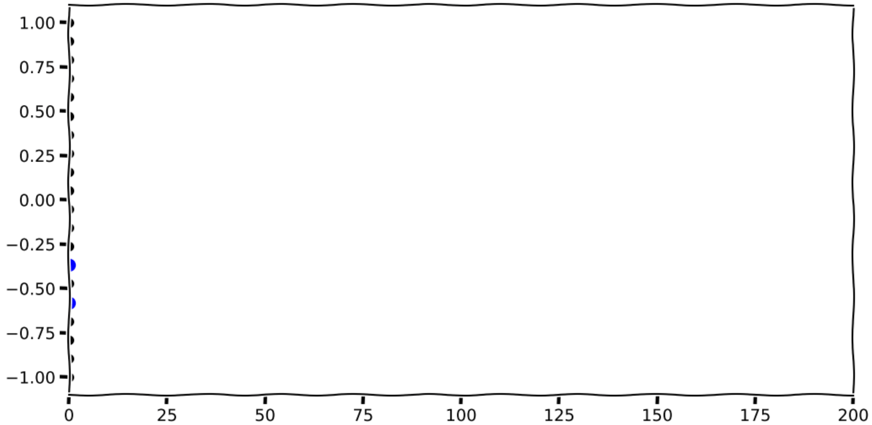




Some take-home messages

- Collective dynamics is a very rich area of mathematics with lots of applications, including in physics, engineering, biology, or the life and social sciences.
- Modelling pedestrian dynamics uses tools from several areas, mainly dynamical systems, but including ODEs, PDEs, Markov processes.
- To analyse these models, we use tools from analysis, probability, and dynamical systems
- There is also a lot of work on controlling these models (e.g. reaching consensus on the same opinion, birds flying in the same direction, evacuating a room).

WARWICK
THE UNIVERSITY OF WARWICK



Thank you for your attention!

Any questions?

You can find out more about me and my research on my website and social media:

https://warwick.ac.uk/fac/sci/maths/people/staff/gomes/

WARWICK
THE UNIVERSITY OF WARWICK